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EFFECTS OF GRAVITY ON RADIALLY SUPPORTED TELESCOPIC MIRRORS

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Abstract—The paper presents an analysis of the optically effective distortions of a telescopic mirror due to gravity loads, when the mirror is positioned so that the optical axis is horizontal.

The mirror is considered a shallow paraboloidal shell of variable thickness with antisymmetric gravity loads, supported on its outer rim by arbitrarily distributed radial forces.

1. INTRODUCTION

THE PURPOSE of this investigation is the determination of the small, elastic deflections of mirrors for large telescopes due to gravity effects. As indicated in Fig. 1, such mirrors have a flat back and a paraboloidal reflecting surface. For analytical purposes, such mirrors may be considered shallow paraboloidal shells of variable thickness. For typical mirrors, the total thickness variation ranges from 10% to 20%, and the diameter to thickness ratio

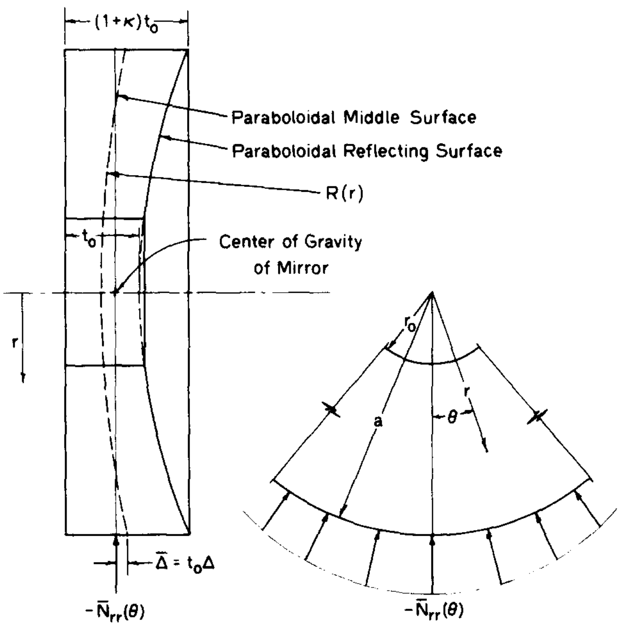

FIG. 1. Radially supported telescopic mirror.

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varies in the range of 7 to 10. In the range indicated the rise of the middle surface is only 1% to 2% of the diameter, so that simplifications based on this extreme shallowness can be made in the analysis. For the case of uniform thickness, the different equations for spherical shells are given in [1] for loads without rotational symmetry. A much simpler set of equations for shallow spherical shells is derived in [2], using the premise of shallowness to replace the spherical surface, as an approximation, by a paraboloid of revolution. A generalization of this analysis to paraboloidal shells of variable thickness is therefore better suited for the study of mirrors than a similar generalization of [1].

The equations for shallow shells of uniform thickness have been studied extensively for various loadings and boundary conditions [3–6]. Flat plates of variable thickness have been studied [7], and also certain classes of shallow shells of variable thickness for cases of rotational symmetry [8], and antisymmetry [9].

A study of telescopic mirrors without central holes has been made in [10] on an approximate basis, treating the mirror as a flat plate with corrective loads to allow for the curvature. However, the solution obtained in [10] violates compatibility requirements for a flat plate of variable thickness. Further, it is shown in [11], that the flat place concept as used in [10] leads to incorrect relations for the determination of the displacements of the optical surface of the mirror. According to [11], the consequences of the various approximations vary depending on the parameters and on the type of mirror support and may result in severe errors.

The analysis presented here, valid for paraboloidal thickness variation, is a special case of the analysis for arbitrary thickness variation in [12], an analysis which follows the approach of [2], but is valid for shells with a central hole and applied loads not derivable from a potential function. Displacements tangent to the middle surface of the shell are also obtained since they are needed to determine the optically effective distortions of the reflecting surface of the mirror. Finally, an estimate of the effects of shear deformation is developed and included in the results presented.

The analysis has been used to compute tables for the determination of optically effective deformations of telescopic mirrors on radial supports [13].

2. DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS FOR TELESCOPIC MIRRORS ON RADIAL SUPPORTS

The differential equations and stress boundary conditions of [12], valid for elastic, homogeneous, and isotropic shallow shells of nonuniform thickness, are presented here for the special case of a telescopic mirror with a parabolic thickness variation, subjected to gravity loads, and radially supported on its outer rim.

Let a be the outer rim radius, t_0 and R_0 be the thickness and radius of curvature of the middle surface at the center of the mirror,† respectively, and let E be Young's modulus, and ν be Poisson's ratio. Then define the following quantities, $\bar{N}_{rr}$, $\bar{N}_{\theta\theta}$, and $\bar{N}_{r\theta}$, the direct stresses tangent to the middle surface; $\bar{V}_r$ and $\bar{V}_\theta$, the transverse shear stress resultants; $\bar{M}_{rr}$, $\bar{M}_{\theta\theta}$, and $\bar{M}_{r\theta}$, the stress couples; $\bar{p}_r$, $\bar{p}_\theta$, and $\bar{p}$, components of the load intensity acting in and normal to the tangent plane to the middle surface; $\bar{u}$, $\bar{v}$ and $\bar{w}$, the components of the displacement of the middle surface in the meridional, circumferential, and normal directions, respectively (see Fig. 2); and finally the thickness t, a function of the radius r.

† If the mirror has a central hole, t_0 and R_0 are values of the functions specifying the thickness and radius of curvature, respectively, at the nominal center of the mirror, Fig. 1.

After introduction of the dimensionless quantities

$$\rho = \frac{r}{a}, \quad (\ )' = \frac{\mathrm{d}}{\mathrm{d}\rho}, \quad \alpha = \frac{a}{R_0}, \quad \beta = \frac{t_0}{a}, \quad t = \frac{\bar{t}}{t_0},$$

$$w = \frac{\bar{w}}{a}, \quad u = \frac{\bar{u}}{a}, \quad v = \frac{\bar{v}}{a}, \quad p_r = \frac{a\bar{p}_r}{t_0 E}, \quad p_\theta = \frac{a\bar{p}_\theta}{t_0 E}, \quad p = \frac{a\bar{p}}{t_0 E}$$

$$N_{rr} = \frac{\bar{N}_{rr}}{t_0 E}, \quad N_{\theta\theta} = \frac{\bar{N}_{\theta\theta}}{t_0 E}, \quad N_{r\theta} = \frac{\bar{N}_{r\theta}}{t_0 E}, \quad V_r = \frac{\bar{V}_r}{t_0 E}, \quad V_\theta = \frac{\bar{V}_\theta}{t_0 E},$$

$$Q_r = \frac{\bar{Q}_r}{t_0 E}, \quad M_{rr} = \frac{\bar{M}_{rr}}{a t_0 E}, \quad M_{\theta\theta} = \frac{\bar{M}_{\theta\theta}}{a t_0 E}, \quad M_{r\theta} = \frac{\bar{M}_{r\theta}}{a t_0 E}$$

all unknown quantities and the applied loads are expanded into Fourier series. For the intended application, these quantities are even or odd functions of θ,

$$(w, u, N_{rr}, N_{\theta\theta}, V_r, M_{rr}, M_{\theta\theta}, p_r, p)$$
$$= \sum_{n=0}^{\infty} (w_n, u_n, N_{rr_n}, N_{\theta\theta_n}, V_{r_n}, M_{rr_n}, M_{\theta\theta_n}, p_{r_n}, p_n) \cos n\theta$$

and

$$(v, N_{r\theta}, V_\theta, M_{r\theta}, p_\theta) = \sum_{n=0}^{\infty} (v_n, N_{r\theta_n}, V_{\theta_n}, M_{r\theta_n}, p_{\theta_n}) \sin n\theta$$

where all quantities with the subscript n are a function of ρ only. For rotational symmetry, $n = 0$,

$$v_0 = N_{r\theta_0} = V_{\theta_0} = M_{r\theta_0} = p_{\theta_0} = 0.$$

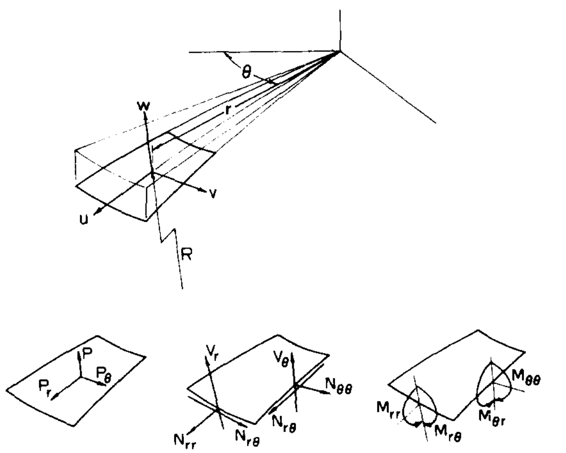

FIG. 2. Deflections, loads, transverse shear and tangential stress resultants, and bending moments.

For the case of the mirror positioned so the optical axis is horizontal, the only non-vanishing load components are for $n = 1$.

The mirror thickness is given by the parabolic law

$$t = 1 + \kappa\rho^2$$

where κ describes the variation, so the load components are then given by

$$p_{r_1} = \frac{Sa}{E}(1+\kappa\rho^2)$$

$$p_{\theta_1} = -\frac{Sa}{E}(1+\kappa\rho^2)$$

$$p_1 = \frac{Sa}{E}\rho\alpha(1+\kappa\rho^2)$$

where S is the specific weight of the mirror material.

Following [2], the direct stresses N_{rr_n}, $N_{\theta\theta_n}$, and $N_{r\theta_n}$ are expressed in [12] by means of a stress function. The expressions are, for $n = 0$,

$$N_{rr_0} = \frac{1}{\rho}\phi_0$$

$$N_{\theta\theta_0} = \phi_0'$$

where ϕ_0 is a stress function, and for $n = 1$,

$$N_{rr_1} = \phi_1 + \frac{c_1}{\rho} - \frac{Sa}{E}\left(\rho + \frac{\kappa}{2}\rho^3\right)$$

$$N_{\theta\theta_1} = \rho\phi_1' + 2\phi_1 - \frac{Sa}{E}(\rho+\kappa\rho^3)$$

$$N_{r\theta_1} = \phi_1$$

where ϕ_1 is a stress function and c_1 is an arbitrary constant.

For values $n \geq 2$,

$$N_{rr_n} = \frac{1}{\rho}F_n' - \frac{n^2}{\rho^2}F_n$$

$$N_{\theta\theta n} = F_n''$$

$$N_{r\theta_n} = n\left(\frac{F_n}{\rho}\right)'$$

where F_n is a stress function.

As a matter of convenience, the thickness $t(\rho)$ which appears explicitly in the differential equations and boundary conditions in [12] is specified approximately by

$$t = e^{(\gamma/2)\rho^2}$$

a simplification used previously in [7]. The value γ is chosen so that the errors introduced by this approximation are minimized according to a mean square law, giving the relation

$$\gamma = 2\kappa - \tfrac{3}{4}\kappa^2.$$

The equations applicable for $n = 0$ are

$$\psi_0'' + \left(\frac{1}{\rho} + 3\gamma\rho\right)\psi_0' + \left(-\frac{1}{\rho^2} + 3\gamma\nu\right)\psi_0 = -\frac{12(1-\nu^2)}{\beta^2} e^{-(3\gamma/2)\rho^2}\left(\alpha\phi_0 + \frac{c_0}{\rho}\right) \tag{2.1a}$$

$$\phi_0'' + \left(\frac{1}{\rho} - \gamma\rho\right)\phi_0' + \left(-\frac{1}{\rho^2} + \gamma\nu\right)\phi_0 = \alpha\, e^{(\gamma/2)\rho^2}\psi_0 \tag{2.1b}$$

where

$$\psi_0 = w_0' \tag{2.2}$$

and c_0 is an arbitrary constant. Equations (2.1) form a fourth order system of simultaneous ordinary differential equations. However, the arbitrary constant c_0 appears in (2.1a), so that a total of five boundary conditions is required to obtain the unknown functions ϕ_0 and ψ_0. For the case of stress boundary conditions for a shell with a central hole, the three quantities

$$N_{rr_0} = \frac{1}{\rho}\phi_0$$

$$M_{rr_0} = -\frac{\beta^2\, e^{(3\gamma/2)\rho^2}}{12(1-\nu^2)}\left(\psi_0' + \frac{\nu}{\rho}\psi_0\right)$$

$$V_{r_0} = -\frac{\beta^2\, e^{(3\gamma/2)\rho^2}}{12(1-\nu^2)}\left[\psi_0'' + \left(\frac{1}{\rho} + 3\gamma\rho\right)\psi_0' + \left(-\frac{1}{\rho^2} + 3\gamma\nu\right)\psi_0\right]$$

are prescribed on one rim. On the other rim, M_{rr_0} and only one of the two quantities N_{rr_0}, V_{r_0} can be prescribed, the remaining one being defined by equilibrium. There are then five relations as required. For a closed shell finite values of N_{rr_0} are obtained at $\rho = 0$ only if

$$\phi_0(0) = 0$$

while continuity of the radial slope ψ_0 requires

$$\psi_0(0) = 0$$

These two conditions, together with specification of M_{rr_0} and one of the two quantities N_{rr_0} or V_{r_0} on the outer rim define the solution.

Knowledge of the function ψ_0 permits the determination of the deflection w_0 by integration of equation (2.2), requiring one prescribed value of w_0. The deflection u_0 may then be obtained without further integration from the relation

$$u_o = e^{-(\gamma/2)\rho^2}(\rho\phi_0' - \nu\phi_0) - \alpha\rho w_0.$$

For $n = 1$, the required equations are

$$\psi_1'' + \left(\frac{3}{\rho} + 3\gamma\rho\right)\psi_1' + \left(-\frac{3}{\rho^2} + 3\gamma(2+\nu)\right)\psi_1 = -\frac{12(1-\nu^2)}{\beta^2} e^{-(3\gamma/2)\rho^2}$$

$$\times\left[\alpha\phi_1 + \frac{\alpha c_1}{2\rho} + \frac{\bar{c}_1}{\rho^3} - \frac{Sa}{E}\alpha\left(\frac{3}{4}\rho + \frac{5}{12}\kappa\rho^3\right)\right] \tag{2.3a}$$

$$\phi_1'' + \left(\frac{3}{\rho} - \gamma\rho\right)\phi_1' + \left[-\frac{3}{\rho^2} - \gamma(2-\nu)\right]\phi_1 = \alpha\,e^{(\gamma/2)\rho^2}\psi_1$$

$$+\frac{Sa}{E}\left\{\left[\frac{5-\nu}{2}\kappa - (1-\nu)\gamma\right]\rho - \gamma\kappa\left(1-\frac{\nu}{2}\right)\rho^3 + \left(\frac{1-\nu}{\rho^3} - \frac{\nu\gamma}{\rho}\right)c_1\right\} \tag{2.3b}$$

where

$$\psi_1 = \left(\frac{w_1}{\rho}\right)' \tag{2.4}$$

and c_1 and $\bar{c}_1$ are arbitrary constants.

Equations (2.3) form a fourth order system of simultaneous ordinary differential equations. Due to the appearance of the two additional arbitrary constants, a total of six boundary conditions is required for determination of the functions ψ_1 and ϕ_1. For stress boundary conditions for a shell with a central hole, the quantities

$$N_{rr_1} = \phi_1 + \frac{c_1}{\rho} - \frac{Sa}{E}\left(\rho + \frac{\kappa}{2}\rho^3\right)$$

$$N_{r\theta_1} = \phi_1$$

$$M_{rr_1} = -\frac{\beta^2\,e^{(3\gamma/2)\rho^2}}{12(1-\nu^2)}\left[\psi_1' + \frac{(2+\nu)}{\rho}\psi_1\right]$$

$$Q_{r_1} \equiv V_{r_1} + \frac{1}{\rho}M_{r\theta_1}$$

$$= -\frac{\beta^2\,e^{(3\gamma/2)\rho^2}}{12(1-\nu^2)}\left[\psi_1'' + \left(\frac{4}{\rho} + 3\gamma\rho\right)\psi_1' + \left(-\frac{1-\nu}{\rho^2} + 3(2+\nu)\gamma\right)\psi_1\right]$$

which enter the boundary conditions at each of the two rims are not independent. Six values may be specified because the remaining two, one value N_{rr_1} or $N_{r\theta_1}$ and one value M_{rr_1} or Q_{r_1} are defined by equilibrium.

For the closed shell, finite values for N_{rr_1} are obtained only if

$$c_1 = 0, \quad |\phi_1(0)| < \infty$$

while finite values for M_{rr_1} require

$$|\psi_1(0)| < \infty.$$

These three conditions and specification of one value N_{rr_1} or $N_{r\theta_1}$ and one value M_{rr_1} or Q_{r_1} on the outer rim define the solution. Knowledge of ψ_1 permits determination of w_1 by integration of equation (2.4), requiring a prescribed value of w_1 at a point $\bar{\rho}$. The deflection u_1 is then obtained by integration of

$$u_1 = e^{-(\gamma/2)\rho^2}\left[-\nu\rho\phi_1' + (1-2\nu)\phi_1 + \frac{c_1}{\rho} - (1-\nu)\frac{Sa}{E}\rho - (1-2\nu)\frac{Sa}{2E}\kappa\rho^3\right] - \alpha w_1$$

requiring one prescribed value of u_1. The displacement v_1 may then be obtained without integration from

$$v_1 = \rho\,e^{-(\gamma/2)\rho^2}\left[\rho\phi_1' + (2-\nu)\phi_1 - \frac{\nu c_1}{\rho} - (1-\nu)\frac{Sa}{E}\rho - (2-\nu)\frac{Sa}{2E}\kappa\rho^3\right] - u_1 - \alpha\rho w_1.$$

The equations for $n \geq 2$ are

$$w_n'''' + \left[\frac{2}{\rho} + 6\gamma\rho\right] w_n''' + \left[-\frac{1+2n^2}{\rho^2} + 3(3+\nu)\gamma + 9\gamma^2\rho^2\right] w_n''$$
$$+ \left[\frac{1+2n^2}{\rho^3} - \frac{3\gamma(1+2n^2-\nu)}{\rho} + 9\gamma^2\nu\rho\right] w_n' + \left[\frac{n^4-4n^2}{\rho^4} - \frac{3n^2\gamma(3-\nu)}{\rho^2} - 9\gamma^2\nu n^2\right] w_n$$
$$= -\frac{12(1-\nu^2)}{\beta^2}\alpha\, e^{-(3\gamma/2)\rho^2}\left[F_n'' + \frac{1}{\rho}F_n' - \frac{n^2}{\rho^2}F_n\right] \tag{2.5a}$$

and

$$F_n'''' + \left[\frac{2}{\rho} - 2\gamma\rho\right] F_n''' + \left[-\frac{1+2n^2}{\rho^2} - (3-\nu)\gamma + \gamma^2\rho^2\right] F_n'' + \left[\frac{1+2n^2}{\rho^3} + \frac{\gamma(1+2n^2+\nu)}{\rho} - \gamma^2\nu\rho\right] F_n'$$
$$+ \left[\frac{n^4-4n^2}{\rho^4} - \frac{\gamma n^2(3+\nu)}{\rho^2} + \nu\gamma^2 n^2\right] F_n = \alpha\, e^{(\gamma/2)\rho^2}\left[w_n'' + \frac{1}{\rho}w_n' - \frac{n^2}{\rho^2}w_n\right]. \tag{2.5b}$$

The stress boundary conditions for a shell with a central hole require specification of the four quantities.

$$N_{rr_n} = \frac{1}{\rho}F_n' - \frac{n^2}{\rho^2}F_n$$

$$N_{r\theta_n} = \frac{n}{\rho}F_n' - \frac{n}{\rho^2}F_n$$

$$M_{rr_n} = -\frac{\beta^2 e^{(3\gamma/2)\rho^2}}{12(1-\nu^2)}\left[w_n'' + \nu\left(\frac{1}{\rho}w_n' - \frac{n^2}{\rho^2}w_n\right)\right]$$

$$Q_{r_n} \equiv V_{r_n} + \frac{n}{\rho}M_{r\theta_n}$$
$$= -\frac{\beta^2 e^{(3\gamma/2)\rho^2}}{12(1-\nu^2)}\left\{w''' + \left[\frac{1}{\rho} + 3\gamma\rho\right]w'' + \left[-\frac{(1+2n^2-\nu n^2)}{\rho^2} + 3\nu\gamma\right]w_n'\right.$$
$$\left. + \left[\frac{3n^2-\nu n^2}{\rho^3} - \frac{3\nu n^2\gamma}{\rho}\right]w_n\right\}$$

on the inner and outer rims to determine the functions w_n and F_n. For the closed shell finite values for $N_{rr_n}(0)$ and $N_{r\theta_n}(0)$ require

$$F_n(0) = 0$$
$$F_n'(0) = 0$$

while finite values for the bending moments at $\rho = 0$ require

$$w_n(0) = 0$$
$$w_n'(0) = 0.$$

These four conditions and specification of the quantities N_{rr_n}, $N_{r\theta_n}$, M_{rr_n}, and Q_{r_n} at the outer rim provide the required eight conditions necessary to determine w_n and F_n. Knowledge of w_n and F_n permits determination of u_n by integration of

$$u_n' = \mathrm{e}^{-(\gamma/2)\rho^2}\left(-\nu F_n''+\frac{1}{\rho}F_n'-\frac{n^2}{\rho^2}F_n\right)-\alpha w_n.$$

The constant of integration, which is selected as the value of u_n at the interior rim, $u_n(\rho_0)$, is given by the relation

$$u_n(\rho_0) = \frac{\rho_0^2\,\mathrm{e}^{-(\gamma/2)\rho_0^2}}{(n^2-1)}\left\{F_n'''(\rho_0)-\gamma\rho_0F_n''(\rho_0)+\left(\frac{-1+\nu-\nu n^2-2n^2}{\rho_0^2}+\nu\gamma\right)F_n'(\rho_0)\right.$$
$$\left.+\left(\frac{3n^2}{\rho_0^3}-\frac{\nu n^2}{\rho_0}\right)F_n(\rho_0)-\alpha\,\mathrm{e}^{(\gamma/2)\rho_0^2}\left[w_n'(\rho_0)-\frac{1}{\rho_0}w_n(\rho_0)\right]\right\}.$$

The displacement v_n may then be obtained from

$$v_n = \frac{\rho\,\mathrm{e}^{-(\gamma/2)\rho^2}}{n}\left(F_n''-\frac{\nu}{\rho}F_n'-\frac{\nu n^2}{\rho^2}F_n\right)-\frac{u_n}{n}-\frac{\alpha\rho w_n}{n}.$$

The telescopic mirror is radially supported on its outer rim by means of a distributed edge force

$$N_{rr}(\theta) = \frac{\bar{N}_{rr}(\theta)}{t_0E} = \sum_{n=0} N_{rr_n}\cos n\theta \tag{2.6}$$

where the dimensionless values N_{rr_n} depend upon the support system. The supports are such that the resultant of the forces, $a\bar{N}_{rr}(\theta)\,\mathrm{d}\theta$, acting on an element of the rim $ad\theta$, lies in the plane of the center of gravity of the mirror. As shown in Fig. 1, this plane has the eccentricity $\bar{\Delta} = t_0\Delta$ with respect to the middle surface of the mirror at $\rho = 1$.

The forces and moments on the outer rim, $\rho = 1$, can thus be expressed in terms of the values N_{rr_n}

$$M_{rr_n} = \Delta\beta N_{rr_n} \tag{2.7a}$$

$$Q_{r_n} = -\alpha N_{rr_n} \tag{2.7b}$$

$$N_{r\theta_n} = 0. \tag{2.7c}$$

The quantities α, Δ, and the focal length $L = 2af$, f being the aperture number, can be expressed in terms of the major nondimensional parameters κ, β, and $\rho_0 = r_0/a$. The relations are

$$\alpha = \beta\kappa$$

$$\Delta = \frac{\kappa}{4}\left[2-\frac{1+\frac{2}{3}\kappa-\rho_0^4(1+\frac{2}{3}\kappa\rho_0^2)}{1+\frac{1}{2}\kappa-\rho_0^2(1+\frac{1}{2}\kappa\rho_0^2)}\right]$$

$$L = \frac{a}{4\kappa\beta}.$$

To apply the above analysis the values of the coefficients N_{rr_n} in equations (2.6, 7) must be specified, while the value of N_{rr_1}, independent of the support system, is given by

$$N_{rr_1} = -\frac{Sa}{E}P_1$$

where

$$P_1 = 1+\frac{\kappa}{2}-\rho_0^2-\frac{\kappa}{2}\rho_0^4.$$

The values N_{rr_n} for $n \neq 1$ can be expressed by coefficients P_n in the form

$$N_{rr_n} = -\frac{Sa}{E}P_n.$$

The values P_n are, for $n = 0$,

$$P_0 = \frac{P_1}{2}\frac{\int_0^\pi N_{rr}(\theta)\,d\theta}{\int_0^\pi N_{rr}(\theta)\cos\theta\,d\theta}$$

and for n $\geq$ 1,

$$P_n = P_1\frac{\int_0^\pi N_{rr}(\theta)\cos n\theta\,d\theta}{\int_0^\pi N_{rr}(\theta)\cos\theta\,d\theta}$$

In the special case where the force $N_{rr}(\theta)$ is distributed as $\cos\theta$ over the lower half of the circumference, $-\pi/2 \leq \theta \leq \pi/2$, the values P_n are

$$P_0 = \frac{2}{\pi}P_1 \qquad P_1 = P_1 \qquad P_2 = \frac{4}{3\pi}P_1$$

$$P_3 = 0 \qquad P_4 = -\frac{4}{15\pi}P_1 \qquad P_5 = 0$$

$$P_6 = \frac{4}{35\pi}P_1 \qquad P_7 = 0 \qquad P_8 = -\frac{4}{63\pi}P_1.$$

While the details depend on the support conditions, the Fourier coefficients P_n always decrease somewhat faster than $1/n$. The deformations corresponding to individual values $P_n = 1$, for $n > 3$, also decrease with n, but much faster. As a result, the contribution to the total deformation due to the terms $n > 3$ is found to be always minor, the major terms being $n = 0, 1, 2$ or 3. This fact will be utilized in the next section, when discussing the effect of approximations. The relative unimportance of the terms $n > 3$ is very apparent in Table 1 given in connection with the example in Section 4.

TABLE 1. MEAN DEFLECTIONS† IN CENTIMETERS

n	0	1	2	3	4	5	6
$10^5 \times (w_{sn})_{mean}$	0·301	0·087	2·140	0	0·056	0	0·013

† These values may be compared with the wavelength of visible light, $\lambda \sim 5{\cdot}5 \times 10^{-5}$ cm.

3. DEFORMATION OF THE MIRROR SURFACE

The optical effect of the deformation of the mirror depends upon the normal component of the displacements of the mirror surface w_s. The nth term of its Fourier expansion, w_{sn}, may be computed from the displacements w_n and u_n of the middle surface, and from the effects of the associated direct stresses N_{rr_n} and $N_{\theta\theta_n}$ in accordance with the underlying assumptions of the shell theory. Due to the linearity of the problem, the contributions to the normal displacement of the mirror surface may be considered individually and then added. The first and major contribution is due to w_n, the displacement component normal to the middle surface. Since the angle between the normal to the middle surface and the normal to the mirror surface is small, the contribution from this source is simply w_n.

Due to the parabolic thickness of the mirror, a radial motion u_n of a point on the middle surface results (to first order) in a normal contribution

$$w_n^{(1)} = \beta\kappa\rho u_n. \tag{3.1a}$$

Further, shell theory implies a state of plane stress associated with the resultants N_{rr} and $N_{\theta\theta}$. Changes in the thickness due to the Poisson's ratio effects result therefore in a contribution

$$w_n^{(2)} = \frac{\nu\beta}{2}(N_{rr_n}+N_{\theta\theta_n}). \tag{3.1b}$$

It can be shown ([12], Appendix D) that these expressions can be expected to be accurate to better than 2% for $n \leq 4$. For larger values of n the errors increase, being nearly 8% for $n = 8$. For normal support conditions the convergence of the Fourier series is very fast, so that the contribution due to the terms $n > 3$ is very small and no significant error occurs.

A second Poisson's effect exists in connection with the linearly distributed bending stresses, but this effect and also that due to the rotation of the cross section can be shown to be insignificant [12].

It is shown in the Appendix that for the problem of a telescopic mirror supported on its rim, Fig. 1, the knowledge of the bending displacement permits a simple estimate of the additional effects of shear deformation. The correction due to this effect is

$$w_n^{(3)} = -\frac{\beta^2}{5(1-\nu)}(1+\kappa/2)^2\left(w_n''+\frac{1}{\rho}w_n'-\frac{n^2}{\rho^2}w_n\right) \tag{3.1c}$$

and can be conveniently combined with the contributions given by equations (3.1a, b). The total normal displacement w_{sn} of the surface of the mirror becomes then

$$w_{sn} = w_n+w_n^{(1)}+w_n^{(2)}+w_n^{(3)}. \tag{3.2}$$

In the case of rotational symmetry, $n = 0$, the deflection w_0, and thus w_{s0}, contains an arbitrary constant b_0, a rigid body translation of the mirror. Further, the optical effects of the deflection can be reduced by changing the location of the focus. These two matters, considered together, are equivalent to adding a suitable expression $b_0+b_1\rho^2$ to the computed displacement w_{s0}. If the manner of operation of the telescope permits this refocusing, the values b_0 and b_1 may be selected to minimize optical distortions. This consideration has been included in all results presented, where b_0 and b_1 have been selected to minimize the mean square error.†

In the case of antisymmetry, $n = 1$, the displacement contains an arbitrary term $b_2\rho \cos\theta$ which corresponds to a rigid body rotation of the mirror. In all results given it is again assumed that the most favorable rotation is selected and that the mean square error† is minimized.

4. NUMERICAL EXAMPLE AND DISCUSSION

As a typical example, a quartz mirror with the following dimensions is considered: radius $a = 100$ in., nominal minimum thickness $t_0 = 20$ in., maximum thickness $\bar{t}(1) = 23{\cdot}74$ in., internal hole radius $r_0 = 20$ in., and focal length $L = 668$ in. The nondimensional parameters are, $\kappa = 0{\cdot}187$, $\beta = 0{\cdot}20$ and $\rho_0 = 0{\cdot}20$, and the properties of the material are, $E = 10{\cdot}6 \times 10^6$ lb/in^2, $\nu = \frac{1}{6}$, and $S = 0{\cdot}0795$lb/in^3.

The mirror is deemed to be supported by compressive radial forces, Fig. 3, distributed proportional to $\cos\theta$ for $-\pi/2 \leq \theta \leq \pi/2$ but zero elsewhere.

Table 1 gives the mean values of the components of the deflection of the mirror surface for $n \leq 6$. It is seen that for this type of support, the astigmatic term $n = 2$ dominates. It is further seen that the terms $n > 2$ are unimportant. The mean value of the surface deflection due to the combined effect of all components is $(w_s)_{\text{mean}} = 2{\cdot}16 \times 10^{-5}$ cm.

Smaller deformations of the mirror can be achieved by using the support condition, Fig. 4, where the distribution of the radial edge force is proportional to $\cos\theta$ over the entire rim. In this case all contributions vanish except the term $n = 1$, which has again the value given in Table 1. The total mean deflection in this situation is $(w_s)_{\text{mean}} = 0{\cdot}87 \times 10^{-6}$ cm. The mean deflection is greatly reduced, to about $\frac{1}{20}$ of the former value.

The numerical results for the example were obtained on an IBM 7094 digital computer which was programmed to generate the deflection w_{sn} in terms of a series expansion of the same form as that encountered in the study of uniform thickness shallow shells, i.e. Bessel

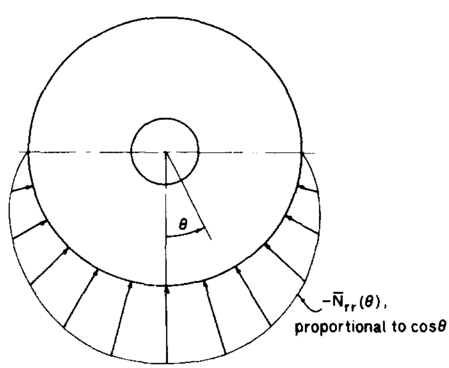

FIG. 3. Distribution of support force on lower half of mirror rim.

† If the deformations of a mirror are a sufficiently small fraction of the wavelength of visible light, the value of the mean square error is a measure of the quality of the mirror. This result follows from an asymptotic analysis for small deviations [15].

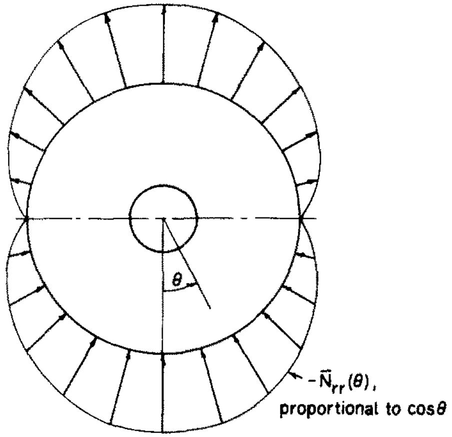

FIG. 4. Cosine support over entire rim.

functions of order n with complex argument. The computer program was arranged to calculate for values $n \leq 6$ twenty terms ρ^{-n}, ρ^{-n+2}, to ρ^{-n+38} and twenty terms $\rho^n \ln \rho$, $\rho^{n+2} \ln \rho$, to $\rho^{n+38} \ln \rho$. The convergence of the series was excellent for all values of the parameters of interest for telescopic mirrors, so that the number of terms carried was more than adequate. A plot of the optically effective deflections at $\theta = \pi/2$, is given in Fig. 5 for the mirror support shown in Fig. 3.

The above results, which allow for the effects of the variation in thickness of the mirror, may be compared with results obtained from [2] for uniform thickness. The values in Table 2 were obtained by using the average thickness to compute an approximate uniform stiffness of the shell, but using the actual nonuniform weight distribution as suggested in [11]. The values in Table 2 also include the shear correction equation (3.1c). Comparison of Tables 1 and 2 shows that the use of the average thickness gives very good results, the differences being everywhere less than 4%. The effect of the variation in stiffness due to

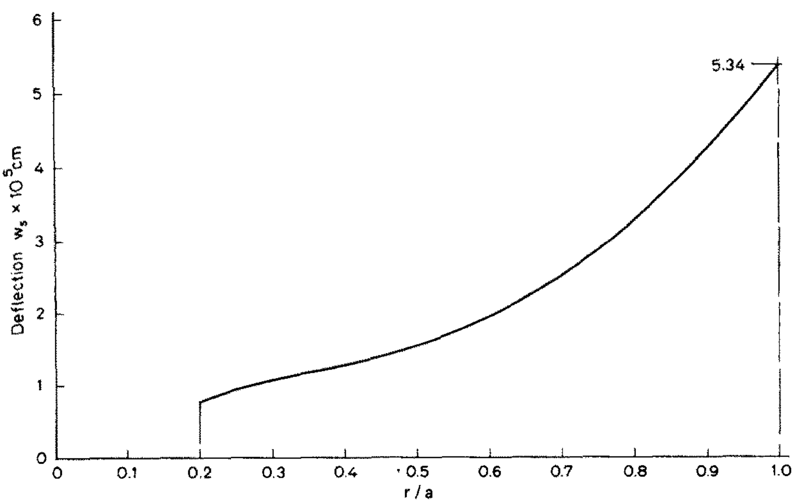

FIG. 5. Optically effective deflection along the line $\theta = \pi/2$ for mirror supported according to Fig. 3.

TABLE 2. MEAN DEFLECTIONS [2] IN CENTIMETERS

n	0	1	2	3	4	5	6
$10^5 \times (w_{sn})_{mean}$	0·290	0·087	2·230	0	0·057	0	0·013

the variation in thickness appears therefore to be small, although somewhat larger deviations have been found for other values of the basic parameters. The use of [2] for engineering purposes appears therefore reasonable, but the inclusion of the shear correction seems desirable.

The effect of gravity on mirrors has recently been studied [16], using the three dimensional theory of elasticity. The analysis is purely numerical, based on finite differences, and gives results in the form of topographical maps of the deflection pattern (not optically corrected). Although a precise comparison of the results of [16] with those of shell theory is thus difficult, the two analyses predict similar deflections on the interior of the mirror. However, the results differ appreciably near the outer rim in locations where radial support forces do not vanish. Such differences are believed to be due to the effect of the concentration of the support forces in [16] on a fraction ($\frac{1}{5}$) of the thickness.

The analysis presented in this paper has been used to prepare tables [13] which permit simple computation of mean surface deflections and rotation at the outer rim of the mirror.

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REFERENCES

[1] W. FLÜGGE, *Stresses in Shells*, Chapter 6. Springer (1960).
[2] E. REISSNER, Stresses and small displacements of shallow spherical shells, *J. Math. Phys.* **25**, 80–85 (1946).
[3] E. REISSNER, On the determination of stresses and displacements for the unsymmetrical deformations of shallow spherical shells, *J. Math. Phys.* **38**, 16–35 (1959).
[4] F. Y. M. WAN, Membrane and bending stresses in shallow spherical shells, *Int. J. Solids Struct.* **3**, 353–366 (1967).
[5] H. REISMANN, G. THURSTON and A. HOLSTON, The shallow spherical shell subjected to point load or hot spot, *Z. angew. Math. Mech.* **45**, 95–103 (1965).
[6] H. REISMANN and P. M. CULKOWSKI, Forced axisymmetric motion of shallow spherical shells, *J. Engng Mech. Div. Am. Soc. Civ. Engrs*, **94**, 653–670 (1968).
[7] O. PILCHER, Die Biegung kreissymmtrischer Platten von veranderlicher Dicke, Berlin, 1928. See *Theory of Plates and Shells*, edited by TIMOSHENKO and WOINOWSKY-KRIEGER, 2nd edition, pp. 298–302. McGraw-Hill (1958).
[8] C. N. DE SILVA and P. M. NAGHDI, Asymptotic solutions of a class of elastic shells of revolution with variable thickness, *Q. appl. Math.* **15**, 182–196 (1969).
[9] T. J. LARDNER and J. G. SIMMONDS, Lateral deformation of shallow shells of revolution, *Int. J. Solids Struct.* **1**, 377–384 (1965).
[10] G. SCHWESINGER, Über günstige Hohlspiegellagerungen hinsichtich der Schwerewirkung senkrecht zur optischen Achse, Doctoral dissertation, Techn. Hochschule, Stuttgart, Germany (1947).
[11] H. H. BLEICH, Analytical determination of the deformations of mirrors on radial supports, Proc. Symp. *Mirror Support Systems for Large Optical Telescopes*. Tuscon (1966).
[12] R. B. NELSON, Deformations of shallow paraboloidal shells of nonuniform thickness, Doctoral Dissertation, Columbia University, New York (1968).
[13] R. B. NELSON and H. H. BLEICH, Tables for the determination of optical effects of gravity on radially supported telescopic mirrors, Columbia University, New York (1968).

[14] R. D. MINDLIN, Influence of rotatory inertia and shear on flexural motions of isotropic elastic plates, *J. appl. Mech.* **18**, 31–38 (1951).
[15] M. BORN and E. WOLF, *Principles of Optics*, 2nd edition revised, pp. 463–464. Macmillan (1964).
[16] A. J. MALVICK and E. T. PEARSON, Theoretical elastic deformations of a 4m diameter optical mirror using dynamic relaxation, *Appl. Optics* 7, 1207–1212 (1968).

APPENDIX

Estimate of shear deformation

The problem of the bending of plates including shear deformation is considered in [14], where the response of a plate is characterized by three quantities, a transverse deflection w, and two angles ψ_r and ψ_θ. The latter are, respectively, the rotation of an elemental area of the middle surface of the plate in the r and θ direction. For the present purpose the relations obtained in [14] are written in polar coordinates, and in the dimensionless variables defined in Section 2.

The moments and shears and the quantities w, ψ_r, ψ_θ are related by

$$M_{rr} = \frac{\beta^2 t^3}{12(1-\nu^2)}\left(\frac{\partial \psi_r}{\partial \rho} + \frac{\nu}{\rho}\frac{\partial \psi_\theta}{\partial \theta}\right) \tag{A.1a}$$

$$M_{\theta\theta} = \frac{\beta^2 t^3}{12(1-\nu^2)}\left(\frac{1}{\rho}\frac{\partial \psi_\theta}{\partial \theta} + \nu\frac{\partial \psi_r}{\partial \rho}\right) \tag{A.1b}$$

$$M_{r\theta} = \frac{\beta^2 t^3}{24(1+\nu)}\left(\frac{1}{\rho}\frac{\partial \psi_r}{\partial \theta} + \frac{\partial \psi_\theta}{\partial \rho}\right) \tag{A.1c}$$

$$V_r = \frac{kt}{2(1+\nu)}\left(\frac{1}{\rho}\frac{\partial w}{\partial \rho} + \psi_r\right) \tag{A.2a}$$

$$V_\theta = \frac{kt}{2(1+\nu)}\left(\frac{1}{\rho}\frac{\partial w}{\partial \theta} + \psi_\theta\right) \tag{A.2b}$$

where t is a dimensionless thickness.

The factor k in equations (A.2) depends upon the distribution of the shear stress. In static problems the distribution is approximately parabolic, in which case k becomes $\frac{5}{6}$. This value is used below. For static problems, the differential equations of equilibrium, obtained from [14] by dropping the inertia terms, are

$$\frac{\partial}{\partial \rho}(\rho M_{rr}) + \frac{\partial}{\partial \theta}M_{r\theta} - M_{\theta\theta} - \rho V_r = 0 \tag{A.3a}$$

$$\frac{\partial}{\partial \rho}(\rho M_{r\theta}) + \frac{\partial}{\partial \theta}M_{\theta\theta} + M_{r\theta} - \rho V_\theta = 0 \tag{A.3b}$$

$$\frac{\partial}{\partial \rho}(\rho V_r) + \frac{\partial}{\partial \theta}V_\theta + \rho p = 0. \tag{A.3c}$$

A special solution suitable for the present purpose is obtained by assuming that ψ_r and ψ_θ may be obtained from a potential function w_B,

$$\psi_r = -\frac{\partial w_B}{\partial \rho} \tag{A.4a}$$

$$\psi_\theta = -\frac{1}{\rho}\frac{\partial w_B}{\partial \theta}. \tag{A.4b}$$

The function w is eliminated from equations (A.1–3) resulting in the equation

$$\nabla^4 w_B = \frac{12(1-\nu^2)}{\beta^2 t^3} p \tag{A.5}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2}.$$

The moments and shear are expressed in terms of w_B

$$M_{rr} = -\frac{\beta^2 t^2}{12(1-\nu^2)}\left[\frac{\partial^2 w_B}{\partial \rho^2} + \nu\left(\frac{1}{\rho}\frac{\partial w_B}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 w_B}{\partial \theta^2}\right)\right] \tag{A.6a}$$

$$M_{\theta\theta} = -\frac{\beta^2 t^3}{12(1-\nu^2)}\left[\frac{1}{\rho}\frac{\partial w_B}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 w_B}{\partial \theta^2} + \nu\frac{\partial^2 w_B}{\partial \rho^2}\right] \tag{A.6b}$$

$$M_{r\theta} = -\frac{\beta^2 t^3}{12(1+\nu)}\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial w_B}{\partial \theta}\right) \tag{A.6c}$$

$$V_r = -\frac{\beta^2 t^3}{12(1-\nu^2)}\frac{\partial}{\partial \rho}(\nabla^2 w_B) \tag{A.7a}$$

$$V_\theta = -\frac{\beta^2 t^3}{12(1-\nu^2)}\frac{1}{\rho}\frac{\partial}{\partial \theta}(\nabla^2 w_B). \tag{A.7b}$$

In the case of stress boundary conditions, the solution w_B of the differential equation (A.5) can be obtained independently of the deflection w because the boundary conditions which follow from equations (A.6, 7) contain only derivatives of w_B. Equations (A.5–7) are of the same form as the corresponding equations in Kirchhoff's plate theory, i.e. w_B is the deflection due to bending effects when shear deformations are neglected. Just as for the conventional plate theory, $M_{r\theta}$ and V_r cannot be specified separately, only the value of the combination can be given,

$$Q_r = V_r + \frac{1}{\rho}\frac{\partial M_{r\theta}}{\partial \theta}.$$

This restriction on the boundary conditions is due to the fact that equations (A.4) are not the most general representations of ψ_r and ψ_θ.

The total deflection w is obtained from w_B by substitution of equations (A.4, 7) into equations (A.2), which give after integration

$$w = w_B - \frac{\beta^2 t^2}{5(1-\nu)}\nabla^2 w_B. \tag{A.8}$$

An arbitrary constant could be added to this expression, but is ignored because such a constant is also contained in the solution for w_B. Equation (A.8) indicates that the additional deflection due to shear is given by the second term, so that the total deflection w may be determined from the deflection w_B.

No such simple approach applied for shells, or for plates of variable thickness. However, if the shells are flat and the variation in the thickness is slight, the second term of equation (A.8) should give a reasonable estimate of the shear deformation if the thickness t in equation (A.8) is interpreted as a mean value. Using the mean value $t = 1+\kappa/2$, the nth term of the Fourier expansion of the corrective term in equation (A.8) is given by equation (3.1c) in Section 3.

The effect of the correction due to shear on the total deformation may be seen from the following table giving the changes in the mean values of the deflections w_{sn} for the example in Section 4.

EFFECT OF SHEAR ON THE MEAN VALUES OF THE DEFLECTION w_{sn}

n	0	1	2	3	4	5	6
Percentage increase	3	1	5	14	20	25	26

The effect of the correction due to shear in the example is quite small, only 5% for the governing term $n = 2$. However, larger effects occur for larger values of β. In addition, the shape of the deflection curve for $n = 1$ and the corresponding mean value of w_{s1} are very sensitive. The 1% difference given in the above table is due to a coincidence† and is not representative of the shear effect for $n = 1$ in general. For a mirror with the same dimensions, but without a hole, or with a hole $\rho_0 = 0{\cdot}3$, the shear effect for $n = 1$ is about +30% and −30% respectively. The inclusion of the effect is thus indicated.

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Абстракт—В работе дается анализ оптически эффективных дисторсий телескопического зеркала под влиянием сил тяжести, вследстве когда зеркало находится в таком положении, что оптическая ось горизонтальная.

Зеркало считается пологой пораболической оболочкой переменной толщины с антисимметриче скими силами тяжести. Оно опирается на своем внешном борте с помощью произвольно распределенных радиальных усилий.

† The sensitivity of the results for $n = 1$ is due to the fact that the normal surface deflection caused by extension of the middle surface, and that caused by bending and associated shear effects are of opposite sign, so that, depending upon the values ρ_0, κ and β, one of the two effects will prevail and dominate the deflection pattern. As a result, the inclusion of shear may increase or decrease the mean deflection, a surprising situation which does not occur for other values of n. The radical changes in deflection shapes may be seen from the plots of w_{s1} in Figs. C2a–c of [12]. The small change of 1% given above, for $n = 1$, is due to the fact that the example lies on the borderline between the two situations which increase or decrease the mean value of w_{s1}.